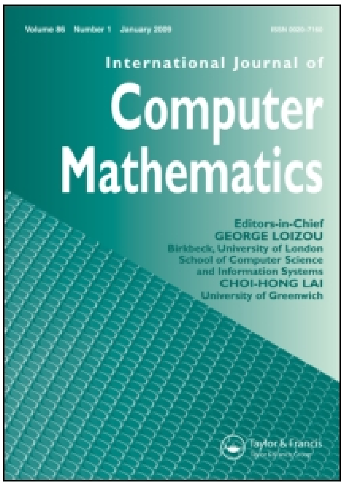

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On the convergence of variational iteration method for nonlinear coupled system of partial differential equations

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In this paper, the sufficient conditions that guarantee the convergence of the variational iteration method when applied to solve a coupled system of nonlinear partial differential equations are presented. Especial attention is given to the error bound of the nth term of the resultant sequence. Numerical examples to show the efficiency of the method are presented.

Keywords: variational iteration method; coupled system of nonlinear partial differential equations; fixed point theorem

2000 AMS Subject Classifications: 65K10; 35A15; 35E99; 68U20; 65G99

1. Introduction

In recent years, much attention has been devoted to the study of the variational iteration method (VIM) given by He (see [10–12] and the references sited therein), for numerically solving a wide range of problems whose mathematical models yield differential equations or a system of differential equations [2,3]. The main reasons for the success of these methods are: there is no need for discretization of the variables and no requirement of large computer memory. Many authors [1,2,4,9,14–23] have pointed out that the VIM can overcome the difficulties arising in the calculation of Adomian's polynomials in Adomian's decomposition method [5–7]. We aim, in this work, to study the convergence of the VIM when applied to solve some nonlinear problems. To illustrate the analysis of the VIM, we limit ourselves to consider the following system of nonlinear equations in the type:

$$L_1u + R_1u + N_1(u) + F_1(u, v) = 0, \tag{1}$$

$$L_2v + R_2v + N_2(v) + F_2(u, v) = 0, \tag{2}$$

with specified initial conditions, where L_i and R_i $(i = 1, 2)$ are linear bounded operators, i.e., it is possible to find numbers $m_i, n_i > 0$ such that $\|L_iu\| \leq m_i\|u\|$, $\|R_iu\| \leq n_i\|u\|$. The nonlinear

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terms $N_1(u)$ and $N_2(v)$ are Lipschitz continuous with $|N_1(u) - N_1(\theta)| < r_1|u - \theta|$ and $|N_2(v) - N_2(\theta)| < r_2|v - \theta|$, $\forall t \in J = [0, T]$. Also, the coupled nonlinear terms $F_i(u, v)$ are Lipschitz continuous with $|F_1(u, v) - F_1(\theta, v)| < s_1|u - \theta|$ and $|F_2(u, v) - F_2(u, \theta)| < s_2|v - \theta|$, $\forall t \in J = [0, T]$. The VIM gives the possibility to write the solution of the Equations (1) and (2) with the aid of the correction functionals:

$$u_p = u_{p-1} + \int_0^t \lambda_1(\tau)[L_1 u_{p-1} + R_1\tilde{u}_{p-1} + N_1(\tilde{u}_{p-1}) + F_1(\tilde{u}_{p-1}, \tilde{v}_{p-1})]\,\mathrm{d}\tau, \tag{3}$$

$$v_p = v_{p-1} + \int_0^t \lambda_2(\tau)[L_2 v_{p-1} + R_2\tilde{v}_{p-1} + N_2(\tilde{v}_{p-1}) + F_2(\tilde{u}_{p-1}, \tilde{v}_{p-1})]\,\mathrm{d}\tau. \tag{4}$$

It is obvious that the successive approximations u_p and v_p, $p > 0$ (the subscript p denotes the pth order approximation), can be established by determining λ_1 and λ_2 general Lagrange multipliers, which can be identified optimally via the variational theory. The functions $\tilde{u}_p$ and $\tilde{v}_p$ are restricted variations, which means that $\delta\tilde{u}_p = \delta\tilde{v}_p = 0$. Therefore, we first determine the Lagrange multipliers that will be identified optimally via integration by parts. The successive approximations u_p and v_p, $p \geq 1$, of the solutions $u(x, t)$ and $v(x, t)$ will be readily obtained upon using the Lagrange multiplier obtained and by using any selective functions u_0 and v_0. Consequently, the exact solution may be obtained by using

$$u = \lim_{p\to\infty} u_p, \quad v = \lim_{p\to\infty} v_p. \tag{5}$$

In what follows, we will apply the VIM to the modified Korteweg-de Vries equation (mKdV) and the coupled system of Burger's equations to illustrate the strength of the method and to establish the exact solutions for these nonlinear problems.

Now, to illustrate how to find the values of the Lagrange multipliers λ_1 and λ_2, we will consider the following case, which depends on the order of the operators L_i $(i = 1, 2)$ in the Equations (1) and (2), and we will study the case of the operators $L_i = \partial/\partial \mathrm{t}$ without loss of generality.

Making the above correction functional stationary, and note that $\delta\tilde{u}_p = \delta\tilde{v}_p = 0$, we obtain

$$\begin{aligned}
\delta u_p &= \delta u_{p-1} + \delta \int_0^t \lambda_1(\tau)\left[\frac{\partial u_{p-1}}{\partial \tau} + R_1\tilde{u}_{p-1} + N_1(\tilde{u}_{p-1}) + F_1(\tilde{u}_{p-1}, \tilde{v}_{p-1})\right]\mathrm{d}\tau \\
&= \delta u_{p-1} + [\lambda_1(\tau)\delta u_{p-1}]_{\tau=t} - \int_0^t \dot{\lambda}_1(\tau)[\delta u_{p-1}]\,\mathrm{d}\tau = 0, \\
\delta v_\mathrm{p} &= \delta v_{p-1} + \delta \int_0^t \lambda_2(\tau)\left[\frac{\partial v_{p-1}}{\partial \tau} + R_2\tilde{v}_{p-1} + N_2(\tilde{v}_{p-1}) + F_2(\tilde{u}_{p-1}, \tilde{v}_{p-1})\right]\mathrm{d}\tau \\
&= \delta v_{p-1} + [\lambda_2(\tau)\delta v_{p-1}]_{\tau=t} - \int_0^t \dot{\lambda}_2(\tau)[\delta v_{p-1}]\,\mathrm{d}\tau = 0,
\end{aligned}$$

where $\delta\tilde{u}_p$ and $\delta\tilde{v}_p$ are considered as restricted variations, i.e. $\delta\tilde{u}_p = \delta\tilde{v}_p = 0$ yields the following stationary conditions:

$$\dot{\lambda}_1(\tau) = 0, \quad 1 + \lambda_1(\tau)|_{\tau=t} = 0, \tag{6}$$

$$\dot{\lambda}_2(\tau) = 0, \quad 1 + \lambda_2(\tau)|_{\tau=t} = 0. \tag{7}$$

The equations in (6) are called Lagrange–Euler equations and the natural boundary condition respectively, the Lagrange multipliers, therefore,

$$\lambda_1(\tau) = \lambda_2(\tau) = -1. \tag{8}$$

Now, the following variational iteration formula can be obtained:

$$u_p = u_{p-1} - \int_0^t [L_1 u_{p-1} + R_1 u_{p-1} + N_1(u_{p-1}) + F_1(u_{p-1}, v_{p-1})]\, d\tau, \tag{9}$$

$$v_p = v_{p-1} - \int_0^t [L_2 v_{p-1} + R_2 v_{p-1} + N_2(v_{p-1}) + F_2(u_{p-1}, v_{p-1})]\, d\tau. \tag{10}$$

We start with an initial approximation, and by using the above iteration formulas (9) and (10), we can directly obtain the other components of the solution.

2. Convergence analysis of the VIM

In this section, we will present some theorems and remarks about the convergence of the VIM. The VIM changes the differential equation to a recurrence sequence of functions. The limit of this sequence is considered as the solution of the partial differential equation.

DEFINITION 1 *A variable quantity v is a functional dependent on a function $u(x)$ if for each function $u(x)$ of a certain class of functions $u(x)$ there corresponds a value v. The variation of a functional $v[u(x)]$ is defined in the following form*:

$$\delta v[u(x)] = \left[\frac{\partial}{\partial \alpha} v[u(x) + \alpha \delta u]\right]_{\alpha=0}. \tag{11}$$

As a well-known result, we have [7].

THEOREM 1 [8] *If a functional $v[u(x)]$ that has a variation achieves a maximum or a minimum at $u = u_0$, where $u(x)$ is an interior point of the domain of definition of the functional, then at $u = u_0$,*

$$\delta v = 0. \tag{12}$$

LEMMA 1 *Let $A : U \to V$ be a bounded linear operator and let $\{u_p\}$ be a convergent sequence in U with a limit u, then $u_p \to u$ in U implies that $A(u_p) \to A(u)$ in V.*

Proof Given the fact that $u_p \to u$.

Now, $\|Au_p - Au\|_V = \|A(u_p - u)\|_V \leq \|A\| \|u_p - u\|_U$.

Hence $\lim_{p\to\infty} \|Au_p - Au\|_V \leq \|A\| \lim_{p\to\infty} \|u_p - u\|_U = 0$, so that $A(u_p) \to A(u)$. ■

THEOREM 2 (uniqueness theorem) *The problems* (1) *and* (2) *has a unique solution whenever* $0 < \alpha_i < 1$, $(i = 1, 2)$ *where,*

$$\alpha_i = (n_i + r_i + s_i)T.$$

Proof Since the solution of the Equations (1) and (2) can take the following forms

$$u = f_1(x) - L_1^{-1}[R_1 u + N_1(u) + F_1(u, v)],$$
$$v = f_2(x) - L_2^{-1}[R_2 v + N_2(v) + F_2(u, v)],$$

where the functions $f_1(x)$ and $f_2(x)$ are the solutions of the homogenous equations $L_1 u = 0$ and $L_2 v = 0$, respectively, and the inverse operators L_i^{-1} are defined by $L_i^{-1}(\bullet) = \int_0^t (\bullet) d\tau$.

Now let, (u, v) and (u^*, v^*) be two different solutions of Equations (1) and (2) then by using the above equations, we get the following:

$$
\begin{aligned}
|u - u^*| &= \left| -\int_0^t [R_1(u - u^*) + N_1(u) - N_1(u^*) + F_1(u, v) - F_1(u^*, v)]\,\mathrm{d}t \right| \\
&\leq \int_0^t [|R_1(u - u^*)| + |N_1(u) - N_1(u^*)| + |F_1(u, v) - F_1(u^*, v)|]\,\mathrm{d}t \\
&\leq (n_1|u - u^*| + r_1|u - u^*| + s_1|u - u^*|)T \\
&\leq \alpha_1|u - u^*|. \\
|v - v^*| &= \left| -\int_0^t [R_2(v - v^*) + N_2(v) - N_2(v^*) + F_2(u, v) - F_2(u, v^*)]\,\mathrm{d}t \right| \\
&\leq \int_0^t [|R_2(v - v^*)| + |N_2(v) - N_2(v^*)| + |F_2(u, v) - F_2(u, v^*)|\,\mathrm{d}t \\
&\leq (n_2|v - v^*| + r_2|v - v^*| + s_2|v - v^*|)T \\
&\leq \alpha_2|v - v^*|,
\end{aligned}
$$

from which we get $(1 - \alpha_1)|u - u^*| \leq 0$, $(1 - \alpha_2)|v - v^*| \leq 0$. Since $0 < \alpha_i < 1$, then $|u - u^*| = 0$, $|v - v^*| = 0$ implies, $u = u^*$ and $v = v^*$and this complete the proof. ■

Now, to prove the convergence of the VIM, we will rewrite Equations (9) and (10) in the operator forms as follows:

$$u_p = A_1[u_{p-1}], \tag{13}$$

$$v_p = A_2[v_{p-1}], \tag{14}$$

where the operators A_i take the following forms:

$$A_1[u] = -\int_0^t [L_1 u + R_1 u + N_1(u) + F_1(u, v)]\,\mathrm{d}\tau, \tag{15}$$

$$A_2[v] = -\int_0^t [L_2 v + R_2 v + N_2(v) + F_2(u, v)]\,\mathrm{d}\tau. \tag{16}$$

THEOREM 3 [Banach's fixed point theorem] (convergence theorem) *Assume that X is a Banach space and $A_i : X \to X$ where $(i = 1, 2)$ are nonlinear mapping, and suppose that*

$$\|A_i[u] - A_i[v]\| \leq \gamma_i \|u - v\|, \quad \forall u, v \in X. \tag{17}$$

For some constants $\gamma_i = \alpha_i + m_i T < 1$. Then A_i have a unique fixed point. Furthermore, the sequences (13) *and* (14) *using the VIM with an arbitrary choice of $u_0, v_0 \in X$ converges to the fixed point of A_i, respectively and*

$$\|u_p - u_q\| \leq \left[\frac{\gamma_1^q}{1 - \gamma_1}\right] \|u_1 - u_0\|, \tag{18}$$

$$\|v_p - v_q\| \leq \left[\frac{\gamma_2^q}{1 - \gamma_2}\right] \|v_1 - v_0\|. \tag{19}$$

Proof Denoting by $(C[J], \|\bullet\|)$ the Banach space of all continuous functions on J with the norm defined by $\|f(t)\| = \max_{t \in J} |f(t)|$.

We prove that the sequence $\{u_p\}$ is a Cauchy sequence in this Banach space

$$
\begin{aligned}
\|u_p - u_q\| &= \max_{t\in J} |u_p - u_q| \\
&= \max_{t\in J} \left| -\int_0^t [L_1(u_{p-1} - u_{q-1}) + R_1(u_{p-1} - u_{q-1}) + N_1(u_{p-1}) \right. \\
&\quad \left. - N_1(u_{q-1}) + F_1(u_{p-1}, v) - F_1(u_{q-1}, v)]\,\mathrm{d}\tau \right| \\
&\le \max_{t\in J} \int_0^t [|L_1(u_{p-1} - u_{q-1})| + |R_1(u_{p-1} - u_{q-1})| \\
&\quad + |N_1(u_{p-1}) - N_1(u_{q-1})| + |F_1(u_{p-1}, v) - F_1(u_{q-1}, v)|]\,\mathrm{d}\tau \\
&\le \max_{t\in J} \int_0^t [(m_1 + n_1 + r_1 + s_1)|u_{p-1} - u_{q-1}|]\,\mathrm{d}\tau \\
&\le \gamma_1 \|u_{p-1} - u_{q-1}\|.
\end{aligned}
$$

Let $p = q + 1$, then

$$\|u_{q+1} - u_q\| \le \gamma_1 \|u_q - u_{q-1}\| \le \gamma_1^2 \|u_{q-1} - u_{q-2}\| \le \cdots \le \gamma_1^q \|u_1 - u_0\|.$$

Hence by the triangle inequality and the formula for the sum of geometric progression, we obtain for $p > q$, we have

$$
\begin{aligned}
\|u_p - u_q\| &\le \|u_{q+1} - u_q\| + \|u_{q+2} - u_{q+1}\| + \cdots + \|u_p - u_{p-1}\| \\
&\le [\gamma_1^q + \gamma_1^{q+1} + \cdots + \gamma_1^{p-1}]\|u_1 - u_0\| \\
&\le \gamma_1^q [1 + \gamma_1 + \gamma_1^2 + \cdots + \gamma_1^{p-q-1}]\|u_1 - u_0\| \\
&\le \gamma_1^q \left[\frac{1 - \gamma_1^{p-q}}{1 - \gamma_1}\right] \|u_1 - u_0\|.
\end{aligned}
$$

Since $0 < \gamma_1 < 1$ so, $1 - \gamma_1^{p-q} < 1$,

$$\|u_p - u_q\| \le \left[\frac{\gamma_1^q}{1 - \gamma_1}\right] \|u_1 - u_0\|.$$

But $\|u_1 - u_0\| < \infty$ so, as $q \to \infty$ then $\|u_p - u_q\| \to 0$. We conclude that $\{u_p\}$ is a Cauchy sequence in $C[J]$, so, the sequence converges. Also, in the same way, we can prove the convergence of the sequence in Equation (14) and obtain the relation (19) and the proof is complete. ■

THEOREM 4 (error estimate theorem) *The maximum absolute errors of the approximate solutions u_p and v_p to problems* (1) *and* (2) *are estimated to be*

$$\max_{t\in J} |u_{\text{exact}} - u_{\text{p}}| < \beta_1, \tag{20}$$

$$\max_{t\in J} |v_{\text{exact}} - v_p| < \beta_2, \tag{21}$$

where

$$\beta_1 = \frac{\gamma_1^q T}{1 - \gamma_1}[(m_1 + n_1)\|u_0\| + h_1 + k_1] \quad \textit{and} \quad \beta_2 = \frac{\gamma_2^q T}{1 - \gamma_2}[(m_2 + n_2)\|v_0\| + h_2 + k_2], \quad \textit{and}$$

$$h_1 = \max_{t\in J} |N_1(u_0)|, \quad h_2 = \max_{t\in J} |N_2(v_0)|, \quad k_i = \max_{t\in J} |F_i(u_0, v_0)|.$$

Proof From Theorem 3 and inequality (18), we obtain as $p \to \infty$ then $u_p \to u_{\text{exact}}$, $v_p \to v_{\text{exact}}$ and

$$\begin{aligned}
\|u_1 - u_0\| &= \max_{t\in J}\left|-\int_0^t [L_1u_0 + R_1u_0 + N_1(u_0) + F_1(u_0, v_0)]\,\mathrm{d}\tau\right| \\
&\le \max_{t\in J}\int_0^t [|L_1u_0| + |R_1u_0| + |N_1(u_0)| + |F_1(u_0, v_0)|]\,\mathrm{d}\tau \\
&\le T[(m_1 + n_1)\|u_0\| + h_1 + k_1]. \\
\|v_1 - v_0\| &= \max_{t\in J}\left|-\int_0^t [L_2v_0 + R_2v_0 + N_2(v_0) + F_2(u_0, v_0)]\,\mathrm{d}\tau\right| \\
&\le \max_{t\in J}\int_0^t [|L_2v_0| + |R_2v_0| + |N_2(v_0)| + |F_2(u_0, v_0)|]\,\mathrm{d}\tau \\
&\le T[(m_2 + n_2)\|v_0\| + h_2 + k_2],
\end{aligned}$$

so, the maximum absolute errors in the interval J are

$$\begin{aligned}
\|u_{\text{exact}} - u_p\| &= \max_{t\in J}|u_{\text{exact}} - u_p| < \beta_1, \\
\|v_{\text{exact}} - v_p\| &= \max_{t\in J}|v_{\text{exact}} - v_p| < \beta_2.
\end{aligned}$$

This completes the proof. ■

The prior error bounds (20) and (21) can be used at the beginning of a calculation for estimating the number of steps necessary to obtain the required accuracy.

3. Numerical test examples

In order to illustrate the performance of the VIM in solving nonlinear partial differential equations and justify the accuracy and efficiency of the method, we consider the following examples.

Example 1 Consider the following homogenous general form of the mKdV equation [13,21]:

$$u_t + 6u^2u_x + u_{xxx} = 0, \tag{22}$$

with an initial condition

$$u(x, 0) = \sqrt{c}\,\mathrm{sech}[k + \sqrt{c}\,x],$$

for all $c \ge 0$, where k is an arbitrary constant. The exact solution of Equation (22) is given by

$$u(x, t) = \sqrt{c}\,\mathrm{sech}[k + \sqrt{c}(x - ct)].$$

To solve Equation (22) by means of the VIM, we construct a correction functional that reads

$$u_{p+1}(x,t) = u_p(x,t) + \int_0^t \lambda(\tau)[u_{p\tau} + 6\tilde{u}_p^2\tilde{u}_{px} + \tilde{u}_{pxxx}]\,\mathrm{d}\tau, \quad p \geq 0. \tag{23}$$

By the same way (in Section 1), the Lagrange multiplier λ, therefore, can be readily identified

$$\lambda(\tau) = -1. \tag{24}$$

Now, the following variational iteration formula can be obtained

$$u_{p+1}(x,t) = u_p(x,t) - \int_0^t [u_{p\tau} + 6u_p^2 u_{px} + u_{pxxx}]\,\mathrm{d}\tau. \tag{25}$$

We start with an initial approximation $u_0(x,t) = u(x,0)$ and by using the above iteration formula (25), we can directly obtain the other components as

$$\begin{aligned}
u_0(x,t) &= \sqrt{c}\,\mathrm{sech}[k+\sqrt{c}x],\\
u_1(x,t) &= u_0(x,t) + c^2 t\,\mathrm{sech}[k+\sqrt{c}\,x]\tanh[k+\sqrt{c}\,x],\\
u_2(x,t) &= u_1(x,t) + \frac{1}{64}c^{3.5}t^2\,\mathrm{sech}^7[k+\sqrt{c}\,x](-14+84c^3t^2\\
&\quad - (17+96c^3t^2)\cosh[2(k+\sqrt{c}\,x)] + 2(-1+6c^3t^2)\cosh[4(k+\sqrt{c}\,x)]\\
&\quad + \cosh[6(k+\sqrt{c}\,x)] - 224c^{1.5}t\sinh[2(k+\sqrt{c}\,x)]\\
&\quad + 48c^{1.5}t\sinh[4(k+\sqrt{c}\,x)]), \ldots
\end{aligned}$$

and so on, in the same manner, the rest of components of the VIM were obtained. In the most cases, the closed form of the solution may be obtained.

In order to numerically verify whether the proposed methodology led to a higher accuracy, we can evaluate the numerical solutions using $p = 2$ terms of Equation (25). Table 1 shows the analytical solution, numerical solution and the absolute error of the value of the time $t = 1.5$. We achieved a very good approximation with the actual solution of the Equation (22) by using two terms only of the iteration equation derived above. It is evident that the overall errors can be made smaller by adding new terms of the iteration formula. The numerical approximation shows a high degree of accuracy and in most cases of $u_p(x,t)$, the p-term approximation is accurate for quite low values of p, and the solutions are very rapidly convergent by utilizing the VIM. The numerical results we obtained justify the advantage of this method, and even in the few terms, approximation is accurate.

Also, the surface Figure 1 presents the error of the solution in the intervals $0 \leq x \leq 10$ and $0 \leq t \leq 3$.

Table 1. Comparison between the exact solution $u(x,t)$ and the approximate solution $u_p(x,t)$.

x	u_p	u_{exaxt}	$\lvert u_p - u_{\text{exact}}\rvert$
0.0	0.00661129	0.00817341	1.56212 e−03
2.0	0.000894721	0.00110617	2.11447 e−04
4.0	0.000121087	0.00014970	2.86163 e−05
6.0	0.000016387	0.00002026	3.87280 e−06
8.0	2.21779 e−06	2.74192 e−06	5.24126 e−07
10.0	3.00146 e−07	3.71079 e−07	7.09328 e−08

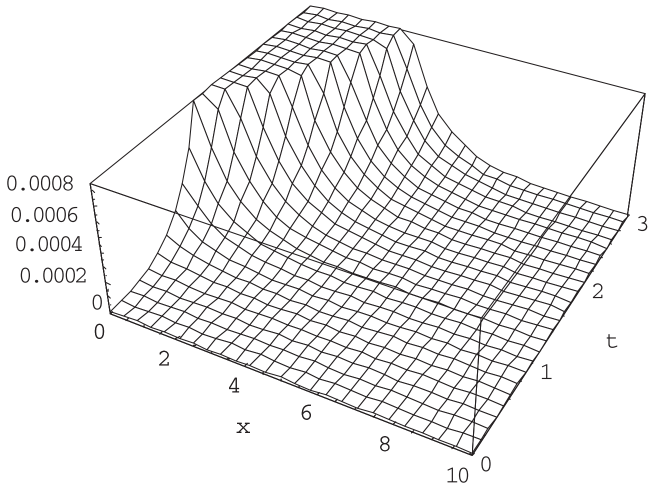


Figure 1. The surface error at $0 \leq x \leq 10$ and $0 \leq t \leq 3$.

It must be noted also that the VIM used here gives the possibility of obtaining an analytical satisfactory solution for which the other techniques of calculation are more laborious and the results contain a great complexity.

Example 2 Consider the following coupled system of Burger's equations in a homogeneous form:

$$u_t - u_{xx} - 2uu_x + (uv)_x = 0, \tag{26}$$

$$v_t - v_{xx} - 2vv_x + (uv)_x = 0. \tag{27}$$

Subject to the following initial conditions

$$u(x,0) = v(x,0) = \sin(x). \tag{28}$$

Equations (26) and (27) are the same as Equations (1) and (2) where $L_i = \partial/\partial t$ and $R_i = \partial^2/\partial x^2$, the nonlinear terms $N_1(u) = -2uu_x$ and $N_2(v) = -2vv_x$, and the coupled nonlinear terms $F_i(u,v) = (uv)_x$, $(i = 1, 2)$.

To solve the Equations (26) and (27) by means of the VIM, we construct a correction functional which reads as follows:

$$u_{p+1}(x,t) = u_p(x,t) + \int_0^t \lambda_1(\tau)[u_{p\tau} - \tilde{u}_{pxx} - 2\tilde{u}_p\tilde{u}_{px} + (\tilde{u}_p\tilde{v}_p)_x]\,\mathrm{d}\tau, \quad p \geq 0. \tag{29}$$

$$v_{p+1}(x,t) = v_p(x,t) + \int_0^t \lambda_2(\tau)[v_{p\tau} - \tilde{v}_{pxx} - 2\tilde{v}_p\tilde{v}_{px} + (\tilde{u}_p\tilde{v}_p)_x]\,\mathrm{d}\tau, \quad p \geq 0. \tag{30}$$

In the same way, the Lagrange multipliers λ_1 and λ_2, therefore, can be readily identified:

$$\lambda_1(\tau) = \lambda_2(\tau) = -1. \tag{31}$$

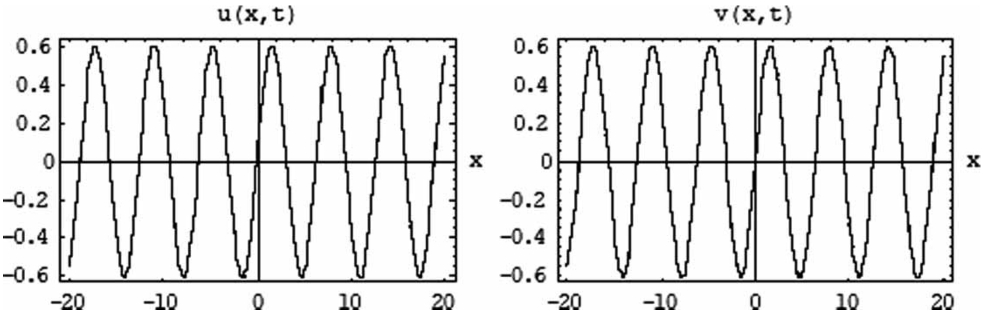
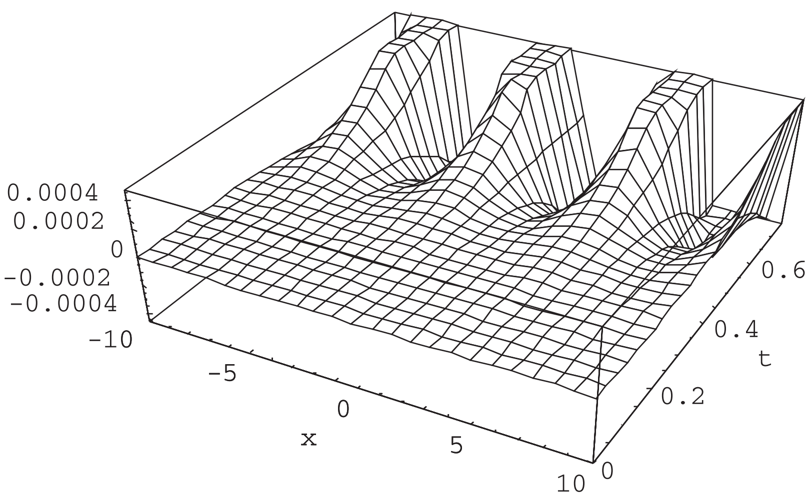


Figure 2. The behaviour of the approximate solutions $u_3(x, t)$ and $v_3(x, t)$ obtained by the VIM for $t = 0.5$ in the interval $-20 \le x \le 20$.



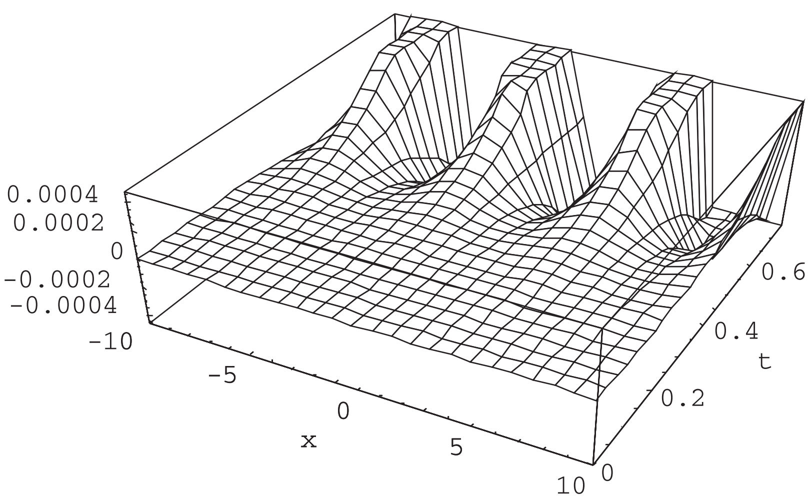


Figure 3. The error of the solutions at $t = 0.75$ in $-10 \le x \le 10$.

Now, the following variational iteration formula can be obtained:

$$u_{p+1}(x,t) = u_p(x,t) - \int_0^t [u_{p\tau} - u_{pxx} - 2u_p u_{px} + (u_p v_p)_x]\,\mathrm{d}\tau,\ p \geq 0. \tag{32}$$

$$v_{p+1}(x,t) = v_p(x,t) - \int_0^t [v_{p\tau} - v_{pxx} - 2v_p v_{px} + (u_p v_p)_x]\,\mathrm{d}\tau,\ p \geq 0. \tag{33}$$

We start with initial approximations $u_0(x,t) = u(x,0)$ and $v_0(x,t) = v(x,0)$ and by using the above iteration formulas (32) and (33), we can directly obtain the other components of the solution:

$$\begin{aligned}
u_0(x,t) &= v_0(x,t) = \sin(x),\\
u_1(x,t) &= v_1(x,t) = (1-t)\sin(x),\\
u_2(x,t) &= v_2(x,t) = \left(1 - t + \frac{t^2}{2!}\right)\sin(x),\\
u_3(x,t) &= v_3(x,t) = \left(1 - t + \frac{t^2}{2!} - \frac{t3}{3!}\right)\sin(x),
\end{aligned}$$

and so on; in the same manner, the rest of components of the VIM were obtained. The closed form of the solutions given by

$$\begin{aligned}
u(x,t) &= \lim_{p\to\infty} u_p(x,t) = \mathrm{e}^{-t}\sin(x),\\
v(x,t) &= \lim_{p\to\infty} v_p(x,t) = \mathrm{e}^{-t}\sin(x).
\end{aligned}$$

This result can be verified through the direct substitution.

The behaviour of the solutions obtained by the VIM (where $p = 3$) is shown for $t = 0.5$ in the interval $-20 \leq x \leq 20$ in Figure 2.

Also, the error of the solutions at $t = 0.75$ in the interval $-10 \leq x \leq 10$ is shown in Figure 3, where the error of the solution $u(x,t)$ is seen in the top of the figure and the error of the solution $v(x,t)$ in the bottom of the figure.

4. Conclusions

In this paper, the convergence analysis of the VIM when applied to solve some nonlinear problems is presented. Moreover, the error bound of the nth term of the resultant sequence is given. The modified KdV equation and the coupled system of Burger's equations are considered here as test examples to illustrate the performance of the VIM in solving nonlinear partial differential equations and to justify the accuracy and efficiency of the method. The main conclusion is that the VIM is a fast convergent method when applied to solve a wide range of nonlinear problems.

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